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On the Symmetry of the Diffusion Coefficient in Asymmetric Simple Exclusion

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We prove the symmetry of the diffusion coefficient that appears in the fluctuation-dissipation theorem for asymmetric simple exclusion processes.

KEY WORDS: Diffusion coefficient; Fluctuation-dissipation theorem; Simple exclusion.

1. INTRODUCTION

The fluctuation-dissipation theorem is the core of Varadhan's method for the analysis of non-gradient systems. In the context of asymmetric simple exclusion processes it was first proved by Landim and Yau[8] (in $d \geqslant 3$) and was subsequently used to establish the diffusive incompressible limit,[2] the first order corrections to the hydrodynamic limit,[3] the equilibrium fluctuations of the density field,[1] and the diffusive hydrodynamic limit when the initial density profile is constant along the direction of the drift.[7] A conceptually similar approach was also used in the derivation of the hydrodynamic limit for the mean-zero asymmetric simple exclusion process.[9]

The content of the theorem is a decomposition of the (normalized) particle currents $(w_i)_{1 \leqslant i \leqslant d}$ (which are not of gradient form) into gradients of the occupation variable and a rapidly fluctuating term. With a suitable interpretation it can be formulated in the following equation:

$$w_i = \sum_{j=1}^{d} D_{ij}(\eta(0) - \eta(e_j)) + Lu_i. \tag{1}$$

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The matrix $D=(D_{ij})_{1\leqslant i,j\leqslant d}$ is called the *diffusion coefficient* and it naturally appears in the PDEs that arise in the hydrodynamic limit. Explicit and variational formulae for D are available[4,5] and it is known to be a smooth function of the particle density.[5] In this article we prove that D is symmetric, thus answering the question raised by Landim, Olla and Yau in ref. 3 and 4.

2. NOTATION AND RESULTS

Let us fix a finite range probability measure $p(\cdot)$ on $\mathbb{Z}^d$, with $p(0)=0$. We denote by L the generator of the simple exclusion process associated to $p(\cdot)$. L acts on local functions on the state space $\mathbb{X}=\{0,1\}^{\mathbb{Z}^d}$ according to:

$$Lf(\xi)=\sum_{x,y} p(y-x)\,\xi(x)\,(1-\xi(y))\,(f(\xi^{x,y})-f(\xi)) \tag{2}$$

where:

$$\xi^{x,y}(z)=\begin{cases}\xi(z) & \text{if } z\neq x,y,\\ \xi(x) & \text{if } z=y,\\ \xi(y) & \text{if } z=x.\end{cases}$$

The symmetric and the anti-symmetric part of $p(\cdot)$ will be denoted by $a(\cdot)$ and $b(\cdot)$ respectively:

$$a(x)=\frac{p(x)+p(-x)}{2}, \qquad b(x)=\frac{p(x)-p(-x)}{2}.$$

In order to avoid degeneracies we will assume that the random walk in $\mathbb{Z}^d$ with one step transition probabilities $a(y-x)$ is irreducible, i.e. $\{x:a(x)>0\}$ generates the group $\mathbb{Z}^d$. An equivalent formulation of this assumption is that the matrix $S=(S_{ij})_{1\leqslant i,j\leqslant d}$ defined by $S_{ij}=\frac{1}{2}\sum p(z)z_i z_j$ is invertible.

The symmetric part of the generator (denoted by L^s) is given by (2) with $p(\cdot)$ replaced by $a(\cdot)$. The measures μ_ρ $(0\leqslant\rho\leqslant 1)$, defined as Bernoulli products of parameter ρ over the sites of $\mathbb{Z}^d$ are invariant under the dynamics. We will denote expectations under μ_ρ by $\langle\cdot\rangle_\rho$ and inner products in $L^2(\mu_\rho)$ by $\langle\cdot,\cdot\rangle_\rho$.

The adjoint of L in $L^2(\mu_\rho)$ is the generator L^* of the simple exclusion process associated to the law $p^*(x)=p(-x)$. Local functions form a core of both L and L^*, and thus L^s extends to a self-adjoint operator in $L^2(\mu_\rho)$.

The particle current along the direction e_i is given by:

$$W_i = \frac{1}{2}\sum_z p(z)z_i\xi(0)(1-\xi(z)) - p(-z)z_i\xi(z)(1-\xi(0)). \tag{3}$$

Equation (1) is to be understood in the Hilbert space of fluctuations, which we define next. Let $\mathcal{G}_\rho$ be the space of local functions g such that:

$$\langle g\rangle_\rho = 0 \qquad \text{and} \qquad \frac{d}{d\theta}\langle g\rangle_\theta\bigg|_{\theta=\rho} = 0.$$

For a $g\in\mathcal{G}_\rho$ we define $\tau_x g = g(\tau_x\xi)$, where $\tau_x\xi(z)=\xi(x+z)$. For any $f\in\mathcal{G}_\rho$ and $i\in\{1,\dots,d\}$ we define:

$$\langle g, f\rangle_{\rho,0} := \sum_x \langle g, \tau_x f\rangle_\rho, \qquad t_i(g) = \langle g, \sum_x x_i\xi(x)\rangle_\rho.$$

Set $\chi(\rho)=\rho(1-\rho)$ and define

$$\begin{aligned}\langle\langle g\rangle\rangle_\rho = & \sup_{\alpha\in\mathbb{R}^d}\left(2\sum_{i=1}^d \alpha_i t_i(g) - \chi(\rho)\alpha\cdot S\alpha\right)\\ & + \sup_{f\in\mathcal{G}_\rho}\left(2\langle g, f\rangle_{\rho,0} - \langle f, (-L^s)f\rangle_{\rho,0}\right).\end{aligned} \tag{4}$$

The Hilbert space of fluctuations $\mathcal{H}(\rho)$ is defined as the closure of $\mathcal{G}_\rho$ under $\langle\langle\cdot\rangle\rangle_\rho^{1/2}$. If we denote by $\mathcal{H}_0$ the space generated by gradients of the occupation variable: $\mathcal{H}_0 = \{\sum\alpha_i(\xi(e_i)-\xi(0));\ \alpha\in\mathbb{R}^d\}$, then by Theorem 1.4 in ref. 8 we have:

$$\mathcal{H}(\rho) = \overline{\mathcal{H}_0 + L\mathcal{G}_\rho}. \tag{5}$$

Notice that unless $\sum zp(z)=0$ the currents W_i do not belong to the space $\mathcal{G}_\rho$. Therefore we define the *normalized* currents $w_i\in\mathcal{G}_\rho$ by:

$$w_i = W_i - \langle W_i\rangle_\rho - (\xi(0)-\rho)\,\frac{d}{d\theta}\langle W_i\rangle_\theta\bigg|_{\theta=\rho}. \tag{6}$$

According to (5) there exist coefficients $(D_{ij})_{1\leqslant i,j\leqslant d}$ (which depend on ρ) such that:

$$w_i - \sum_{j=1}^{d} D_{ij} \times (\xi(0) - \xi(e_j)) \in \overline{L\mathcal{G}_\rho}. \tag{7}$$

The matrix $D = (D_{ij})_{1\leqslant i,j\leqslant d}$ is called the diffusion coefficient of the simple exclusion process. Landim, Olla and Yau proved explicit and variational formulae for D in ref. 4. In the same paper, as well as in ref. 3, the authors question whether there exists a choice of $p(\cdot)$ such that D is asymmetric. The result of this article is the following theorem:

Theorem 1. The diffusion coefficient D defined in (7) is always a symmetric matrix.

3. SOME PROPERTIES OF $\mathcal{H}(\rho)$

In this section we review some properties of the Hilbert space of fluctuations that will be useful in the proof of Theorem 1. We begin with the following lemma.

Lemma 1. If $g \in \mathcal{G}_\rho$ and $h \in \mathbb{Z}^d$, then $\tau_h g = g$ in $\mathcal{H}(\rho)$.

Proof. In view of (4) it suffices to show that:

$$(i)\ \langle \tau_h g - g, f\rangle_{\rho,0} = 0, \forall f \in \mathcal{G}_\rho, \qquad (ii)\ t_i(\tau_h g - g) = 0, i = 1, \dots, d.$$

Using the translation invariance of μ_ρ property (i) follows immediately, while

$$\sum_x \langle \tau_h g - g, x_i \xi(x)\rangle_\rho = h_i \sum_x \langle g, \xi(x)\rangle_\rho.$$

The last expression is trivially zero if $\rho \in \{0, 1\}$, while otherwise by differentiating with respect to θ both sides of the following identity

$$\langle g\rangle_\theta = \int g(\xi) \prod_{x\in \text{supp}(g)} \left(\frac{\theta}{\rho}\right)^{\xi(x)} \left(\frac{1-\theta}{1-\rho}\right)^{1-\xi(x)} d\mu_\rho(\xi),$$

we get

$$\sum_x \langle g, \xi(x)\rangle_\rho = \rho\langle g\rangle_\rho + \chi(\rho)\, \frac{d}{d\theta} \langle g\rangle_\theta \bigg|_{\theta=\rho} = 0, \tag{8}$$

thus establishing (ii). ∎

The following lemma is a generalisation of (5.1) in ref. 4 to the general asymmetric simple exclusion, and can be proved by polarization of (4). The details are left to the reader.

Lemma 2. Let $g, f \in \mathcal{G}_\rho$ and set $\nabla_{e_k}\xi(0) = \xi(0) - \xi(e_k)$ for $k = 1, \dots, d$. Then:

$$(i)\quad \langle\langle \nabla_{e_k}\xi(0), \nabla_{e_\ell}\xi(0)\rangle\rangle_\rho = \chi(\rho)(S^{-1})_{k\ell},$$

$$(ii)\quad \langle\langle \nabla_{e_k}\xi(0), Lg\rangle\rangle_\rho = -\langle\langle \nabla_{e_k}\xi(0), L^* g\rangle\rangle_\rho = \sum_{\ell=1}^{d}(S^{-1})_{k\ell}\langle w_\ell, g\rangle_{\rho,0},$$

$$(iii)\quad \langle\langle \nabla_{e_k}\xi(0), L^s g\rangle\rangle_\rho = 0,$$

$$(iv)\quad \langle\langle L^s g, f\rangle\rangle_\rho = -\langle g, f\rangle_{\rho,0}.$$

4. THE DIFFUSION MATRIX

Recall the definition of the normalised currents w_i given in (3) and (6). It follows by elementary algebra and Lemma 1 that $w_i = W_i^s - h_i$, where

$$W_i^s(\xi) = \frac{1}{2}\sum z_i a(z)(\xi(0) - \xi(z)),$$

is the current of the symmetric simple exclusion with generator L^s, and

$$h_i(\xi) = \sum_z z_i b(z)(\xi(0) - \rho)(\xi(z) - \rho).$$

Hence, the normalized currents for the reversed process are given by $w_i^* = W_i^s + h_i$. Let now $C(\rho)$ (resp. $C_*(\rho)$) be the real vector space generated by the currents $\{w_i; i = 1, \dots, d\}$ (resp. $\{w_i^*; i = 1, \dots, d\}$).

We define the linear operator T (resp. T^*) on $C(\rho) + L\mathcal{G}_\rho$ (resp. $C_*(\rho) + L^*\mathcal{G}\rho$) by:

$$T\Big(\sum_{i=1}^{d}\alpha_i w_i + Lg\Big) = \sum_{i,k=1}^{d}\alpha_i S_{ik}\nabla_{e_k}\xi(0) + L^s g,$$

$$T^*\Big(\sum_{i=1}^{d}\alpha_i w_i^* + L^* g\Big) = \sum_{i,k=1}^{d}\alpha_i S_{ik}\nabla_{e_k}\xi(0) + L^s g.$$

By Theorem 1.4 in ref. 8 we have: $\mathcal{H}(\rho) = \overline{C(\rho) + L\mathcal{G}_\rho} = \overline{C_*(\rho) + L^*\mathcal{G}\rho}$. Now, just as in Lemma 5.4 in ref. 4, T and T^* are norm bounded by 1, hence they can be extended to $\mathcal{H}(\rho)$. Furthermore, it follows easily by computations based on Lemma 2 that T^* is the adjoint of T with respect to $\langle\langle \cdot, \cdot \rangle\rangle_\rho$ and $T^*\nabla_{e_k}\xi(0)$ is orthogonal to $\overline{L\mathcal{G}_\rho}$. Hence by (7) we get

$$\langle\langle w_i, T^*\nabla_{e_k}\xi(0)\rangle\rangle_\rho = \sum_{i=1}^{d} D_{ij} \langle\langle \nabla_{e_j}\xi(0), T^*\nabla_{e_k}\xi(0)\rangle\rangle_\rho,$$

and thus by Lemma 2(i):

$$\chi(\rho) I_d = D \cdot Q,$$

where the matrix $Q = (Q_{jk})_{1 \leqslant j,k \leqslant d}$ is given by:

$$Q_{jk} = \langle\langle T\nabla_{e_j}\xi(0), \nabla_{e_k}\xi(0)\rangle\rangle_\rho.$$

We are now ready to proceed with the proof of Theorem 1.

Proof. (of Theorem 1). Let us denote the reflection operator on $\mathbb{X}$ by

$$R\xi(z) = \xi(-z).$$

The action of R is naturally extended to functions as $Rf(\xi) = f(R\xi)$. Clearly, $R^2 = 1$. Furthermore, the following commutation relation can be readily verified:

$$RL = L^*R. \tag{9}$$

In particular R commutes with L^s and hence, R preserves inner products in $\mathcal{H}(\rho)$.

Notice that W_i^s are anti-symmetric under R, while h_i are R-symmetric. Thus,

$$Rw_i(\xi) = -W_i^s(\xi) - h_i(\xi) = -w_i^*(\xi). \tag{10}$$

It is a direct consequence of (9), (10), and the observation that $R\nabla_{e_k}\xi(0) = -\nabla_{e_k}\xi(0)$ in $\mathcal{H}(\rho)$ that

$$RT = T^*R.$$

Therefore,

$$
\begin{aligned}
Q_{jk} &= \langle\langle RT\nabla_{e_j}\xi(0), R\nabla_{e_k}\xi(0)\rangle\rangle_\rho \\
&= \langle\langle T^* R\nabla_{e_j}\xi(0), R\nabla_{e_k}\xi(0)\rangle\rangle_\rho \\
&= \langle\langle \nabla_{e_j}\xi(0), T\nabla_{e_k}\xi(0)\rangle\rangle_\rho \\
&= Q_{kj}.
\end{aligned}
$$

So Q and thus the diffusion coefficient D are symmetric matrices. ∎

Remarks 1. Even though the fluctuation-dissipation theorem for the general asymmetric simple exclusion process is only valid in $d \geqslant 3$, the argument in the proof of Theorem 1 could still be used to infer the symmetry of the bulk diffusion coefficient for the *mean-zero* asymmetric simple exclusion process, which exhibits diffusive behavior in any dimension.

2. We chose to use the abstract formalism of ref. 4 for the diffusion coefficient, which, as well as the argument presented here, avoids the use of duality techniques. It is interesting to note that the same space reflection considerations can deduce the symmetry of the diffusion coefficient from the explicit formula for it derived in ref. 5 using duality.

3. The importance of Theorem 1 is underlined by a number of known results involving the symmetric part of $D(\rho)$. Let us mention for instance the variational formulae (6.1) for D^s and $(D^{-1})^s$ in ref. 4, or the invariance principle for the position of a second class particle (Theorem 6.2 and its immediate consequence in ref. 6). Evidently, in view of Theorem 1 these results can all be restated in terms of D.

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